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Fractional Curvatures of Equiaffine Curves in Three-Dimensional Affine Space

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Abstract — This paper presents a method for computing the curvatures of equiaffine curves in three-dimensional affine space by utilizing local fractional derivatives. First, the concepts of α-equiaffine arc length and α-equiaffine curvatures are introduced by considering a general local involving conformable derivative, V-derivative, etc. In fractional calculus, equiaffine Frenet formulas and curvatures are reestablished. Then, it presents the relationships between the equiaffine curvatures and α-equiaffine curvatures. Furthermore, graphical representations of equiaffine and α-equiaffine curvatures illustrate their behavior under various conditions.

Keywords *Fractional derivative, equiaffine curvatures, affine space*

Mathematics Subject Classification (2020) 26A33, 53A04

1. Introduction

Differential geometry, investigating the properties of curves, surfaces, and manifolds, provides a profound understanding of mathematical objects' intrinsic characteristics. Recently, the intersection of differential geometry with fractional derivatives has opened up new avenues of research and exploration. Fractional derivatives, which extend the classical notion of derivatives to non-differentiable functions, find application in various fields. This paper deals with the fascinating realm where local fractional derivatives and differential geometry meet, particularly in analyzing equiaffine curves.

The concept of fractional calculus, encompassing fractional derivatives, can be traced back to the endeavors of mathematicians, such as L'Hospital and Leibniz, in the 18th century. During this period, they explored the feasibility of extending the notion of derivatives to non-integer orders. However, their pioneering ideas did not gain widespread acceptance at that time. It was not until the 19th century that significant advancements were made in this field. Notable contributions were made by Cauchy, Weierstrass, and Liouville, who played key roles in shaping the theory of fractional derivatives. Cauchy, for instance, introduced the fundamental concept of fractional derivatives and integrals, laying the groundwork for their mathematical treatment. The efforts of Liouville and Riemann in the mid-19th century were pivotal in advancing the theory of fractional calculus. The Riemann-Liouville fractional derivatives, developed during this era, are extensively employed today. Fractional calculus progressed further throughout the 20th century, with significant contributions from renowned mathematicians; for more details, see [1].

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Beyond its applications in pure mathematics, fractional calculus has gained substantial significance in recent years. In [2], plane curves in equiaffine geometry are examined by considering fractional derivatives. The authors [3] have studied the geometry of curves possessing fractional-order tangent vectors and Frenet-Serret formulas. In [4], the authors obtained some characterizations of curves of fractional order in three-dimensional Euclidean space. The authors [5] provide the Frenet frame compatible with the conformable derivative. In [6], the authors obtained special fractional curve pairs and some characterizations. However, the fractional differential geometry of curves and surfaces is explicitly provided in [7]. In calculus, the chain rule, which holds great importance, is discussed regarding fractional derivatives in [8]. Moreover, fractional derivatives find frequent use in numerical analysis, as evidenced by studies [9, 10]. The applications of fractional calculus extend to various scientific and engineering domains. In the 20th century, fractional derivatives were applied across diverse disciplines, including physics, engineering, signal processing, and control theory, particularly in modeling systems with memory and intricate dynamics. In recent times, fractional calculus has experienced a resurgence of interest, with applications extending to finance, materials science, and bioengineering.

Recent advancements in computational techniques have simplified the handling of fractional derivatives in practical applications. In the present day, fractional calculus is a well-established branch of mathematics, with its applications continuously expanding into diverse scientific and engineering domains. These applications offer valuable tools for modeling and analyzing intricate systems. Notably, these applications span across fields such as medicine [11], bioengineering [12], viscoelasticity [13], and dynamical systems [14, 15]. In contrast to the straightforward expressions of integer-order derivatives and integrals, various more intricate fractional derivatives and integrals exist. Riemann-Liouville, Caputo, and Riesz fractional derivatives are prominent examples of non-local fractional derivatives [16]. On the other hand, conformable [17], truncated M- and V-fractional derivatives [18, 19] represent distinct types of local derivatives.

When dealing with non-local fractional derivatives, the conventional characteristics observed in integer order derivatives, such as the standard Leibniz and chain rules, are not satisfied. The absence of these features presents a significant challenge when formulating a theory of differential geometry, given their crucial importance. To elaborate, opting for a non-local fractional derivative rather than an integer-order derivative constrains the application of techniques derived from Riemannian geometry due to the lack of the Leibniz and chain rules. Many foundational concepts within Riemannian geometry heavily depend on these rules. Consequently, utilizing local fractional derivatives that exhibit the mentioned crucial properties facilitates calculations in differential geometry. For this reason, the study is crafted to focus on incorporating local fractional derivatives.

Our motivation arises from defining a general local fractional derivative operator compatible with all fractional derivatives, as outlined in (2.3). This study investigates equiaffine curves by examining this generalized local fractional derivative. Additionally, we introduce the concepts of equiaffine arc length and curvature. The rationale for considering equiaffine invariants is as follows: If one employs local fractional derivatives to analyze Frenet invariants of a curve, the Frenet frame remains unaffected. However, the utilization of local fractional derivatives impacts the equiaffine Frenet frame. In addition, the α-equiaffine Frenet frame is a new and different frame from the classical Frenet frame. This Frenet frame is obtained by considering the local fractional derivative and is a new study area.

The construction of this paper proceeded as follows: Section 2 provides fundamental concepts related to fractional derivatives. Section 3 discusses equiaffine invariants of equiaffine curves in the 3-dimensional affine space $\mathbb{R}^3$ by considering a general local fractional derivative. Section 4 presents the main results and introduces the α-equiaffine Frenet frame and α-equiaffine curvatures of a curve with equiaffine arc length. Moreover, the section establishes a relationship between the curve's equiaffine curvatures and α-equiaffine curvatures. Besides, it plots graphs of α-equiaffine curvature functions for some values of equiaffine curvatures. Finally, it discusses the need for further research.

2. Preliminaries

This section presents the notion of the V-fractional derivative, initially introduced in [19], due to its wider applicability when contrasted with the local fractional derivatives detailed in Section 1. For a visual depiction, please consult the informative diagram on page 23 of the referenced paper. Consequently, any statement valid for the $\mathcal{V}$-fractional derivative inherently generalizes to the other types of derivatives. Denoting by $\Gamma(\alpha)$ the Euler gamma function defined for the parameter α as described in [20]

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$$

Let $\mathbb{R}^+$ be the set of all positive real numbers. Suppose γ, β, ρ, and δ are complex numbers with positive real components, and p and q are elements of $\mathbb{R}^+$. In this context, we present a six-parameter Mittag–Leffler function denoted as

$${}_i\mathbb{E}^{\rho,\delta,q}_{\gamma,\beta,p}(z) = \sum_{k=0}^{i} \frac{(\rho)_{qk}}{(\delta)_{pk}} \frac{z^k}{\Gamma(\gamma k+\beta)}, z \in \mathbb{C}, Re(z) > 0$$

as detailed in [19]. The function incorporates $\Gamma(\rho)$, representing the gamma function involving ρ, and $(\rho)_{qk}$, which is an extension of the Pochhammer symbol defined by $(\rho)_{qk} = \frac{\Gamma(\rho+qk)}{\Gamma(\rho)}$.

Consider a real number α such that $0 < \alpha \leq 1$, $t \to f(t)$, and $t \in I \subset \mathbb{R}^+$. In this scenario, the truncated $\mathcal{V}$-fractional derivative of $f(t)$ is expressed as

$${}^{\rho}_{i}\mathcal{V}^{\rho,\delta,q}_{\gamma,\beta,p}f(t) = \lim_{\varepsilon\to 0} \frac{f\left(t_i H^{\rho,\delta,q}_{\gamma,\beta,p}(\varepsilon t^{-\alpha}) - f(t)\right)}{\varepsilon} \tag{2.1}$$

where ${}_iH^{\rho,\delta,q}_{\gamma,\beta,p}(\varepsilon t^{-\alpha})(z) = \Gamma(\beta){}_i\mathbb{E}^{\rho,\delta,q}_{\gamma,\beta,p}(z)$.

A function is α-differentiable if the limit in (2.1) exists. Furthermore, the truncated $\mathcal{V}$-fractional derivative operates as a linear operator, following the Leibniz and chain rules, by the principles expounded in [19]

$${}^{\rho}_{i}\mathcal{V}^{\delta,p,q}_{\gamma,\beta,\alpha}f(t) = \frac{t^{1-\alpha}\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}\frac{df(t)}{dt} \tag{2.2}$$

By varying the specific the parameters γ, β, δ, p, and q various other local derivatives such as truncated M-fractional derivatives, alternative and conformable, etc., can be obtained. As indicated in (2.2), the $\mathcal{V}$-fractional derivative maintains a linear relationship with the standard integer order derivative. Consequently, a comprehensive definition of a general local fractional derivative operator is established as follows: Consider a function $C(\alpha, t)$ of class C^4 defined as

$$t \to C(\alpha, t) \in \mathbb{R}^+,\ t \in I \subset \mathbb{R}^+$$

where $C(\alpha, t)$ equals 1 specifically when $\alpha = 1$. Since the coefficient of the term $\frac{df(t)}{dt}$ in (2.2) is function of t, next we generally introduce

$$\frac{d^\alpha}{dt^\alpha} = C(\alpha, t)\frac{d}{dt} \tag{2.3}$$

where as $\frac{d}{dt}$ is the standard derivative operator. For example, (2.3) suggests conformable derivative if $C(\alpha, t) = t^{1-\alpha}$ and a truncated M-fractional derivative if $C(\alpha, t) = \frac{t^{1-\alpha}}{\Gamma(1+\gamma)}$, $(\gamma \in \mathbb{R}^+)$ and etc.

3. Equiaffine Invariants of Space Curves

This section provides brief information from [21–24]. Let $\mathbb{R}^3$ be 3-dimensional affine space and $Mat(3,\mathbb{R})$ be the set of all the square matrices of order 3. We write

$$SL(\mathbb{R}^3) = \{A \in Mat(3,\mathbb{R}) : det(A) = 1\}$$

Then, by an *equiaffine invariant*, we mean an unchanged feature under the actions of $SL(\mathbb{R}^3)$ and the translations of $\mathbb{R}^3$. For example, the volume is an equiaffine invariant. Let $[u_1\ u_2\ u_3]$ is the determinant of vectors $u_1, u_2, u_3 \in \mathbb{R}^3$. Then, the value of $[u_1\ u_2\ u_3]$ is an equiaffine invariant because it measures the volume of parallelopipedon determined by u_1, u_2, and u_3.

If $n = 3$, then the following expressions [24] are written as follows:

Let $t \to y(t)$, $t \in I \subset \mathbb{R}$, is smooth parametrized curve in $\mathbb{R}^3$. $y(t)$ is the *non-degenerate* if, for every $t \in I$,

$$\left[\frac{dy}{dt}(t)\ \ \frac{d^2y}{dt^2}(t)\ \ \frac{d^3y}{dt^3}(t)\right] \neq 0$$

For the sake of simplicity, when we refer to a curve in this paper, we mean a non-degenerate smooth parameterized curve. Subsequently, the *equiaffine arc length function* is defined as

$$\mu(t) = \int^t \left[\frac{dy}{du}(u)\ \ \frac{d^2y}{du^2}(u)\ \ \frac{d^3y}{du^3}(u)\right]^{\frac{1}{6}} du$$

We refer to the curve as being parameterized by *equiaffine arc length* if, for every $\mu \in J \subset \mathbb{R}$,

$$\left[\frac{dy}{d\mu}(\mu)\ \ \frac{d^2y}{d\mu^2}(\mu)\ \ \frac{d^3y}{d\mu^3}(\mu)\right] = 1 \tag{3.1}$$

The set $\left\{\frac{dy}{d\mu}(\mu), \frac{d^2y}{d\mu^2}(\mu), \frac{d^3y}{d\mu^3}(\mu)\right\}$ is referred to as the *equiaffine Frenet frame* of $y(\mu)$. When we take the derivative (3.1) according to the parameter μ, it is apparent that

$$\left[\frac{dy}{d\mu}(\mu)\ \ \frac{d^2y}{d\mu^2}(\mu)\ \ \frac{d^4y}{d\mu^4}(\mu)\right] = 0$$

where the set are linearly dependent for every $\mu \in J$, given by

$$\left\{\frac{dy}{d\mu}(\mu), \frac{d^2y}{d\mu^2}(\mu), \frac{d^4y}{d\mu^4}(\mu)\right\}$$

Therefore, this implies the existence of smooth functions κ and τ on J such that

$$\frac{d^4y}{d\mu^4}(\mu) + \kappa(\mu)\frac{dy}{d\mu}(\mu) + \tau(\mu)\frac{d^2y}{d\mu^2}(\mu) = 0$$

where

$$\kappa(\mu) = -\left[\frac{d^2y}{d\mu^2}(\mu)\ \ \frac{d^3y}{d\mu^3}(\mu)\ \ \frac{d^4y}{d\mu^4}(\mu)\right] \tag{3.2}$$

and

$$\tau(\mu) = \left[\frac{dy}{d\mu}(\mu)\ \ \frac{d^3y}{d\mu^3}(\mu)\ \ \frac{d^4y}{d\mu^4}(\mu)\right] \tag{3.3}$$

The function $\kappa(\mu)$ and $\tau(\mu)$ are called *equiaffine curvature* and *equiaffine torsion* of the curve $y(\mu)$, respectively. The equiaffine curvature and equiaffine torsion are the equiaffine invariants in $\mathbb{R}^3$.

Let y be a non-degenerate smooth curve in $\mathbb{R}^3$ parameterized by equiaffine arc length μ. As a result, the equiaffine equations of Frenet type are presented in matrix form as

$$\begin{bmatrix} \frac{d^2y}{d\mu^2}(\mu) \\ \frac{d^3y}{d\mu^3}(\mu) \\ \frac{d^4y}{d\mu^4}(\mu) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\kappa(\mu) & -\tau(\mu) & 0 \end{bmatrix} \begin{bmatrix} \frac{dy}{d\mu}(\mu) \\ \frac{d^2y}{d\mu^2}(\mu) \\ \frac{d^3y}{d\mu^3}(\mu) \end{bmatrix}$$

4. Fractional Equiaffine Curvatures

This section establish a relationship between the equiaffine curvatures and a curve's α-equiaffine curvatures. It then presents graphical representations of the α-equiaffine curvature functions corresponding to certain equiaffine curvature values.

Proposition 4.1. Let y be a non-degenerate smooth curve in $\mathbb{R}^3$ parameterized by the equiaffine arc length μ. Moreover, let

$$\mu(s) = \int^s (C(\alpha,t))^{-\frac{1}{2}}\, dt \tag{4.1}$$

Then, the parameter s is the equiaffine arc length parameter concerning (2.3). Here, $\mu(s)$ is called α-equiaffine arc length.

Proof. Using (2.3), we have

$$\frac{d^\alpha y}{ds^\alpha} = C(\alpha,s)\frac{dy}{d\mu}\frac{d\mu}{ds}$$

or

$$\frac{d^\alpha y}{ds^\alpha} = (C(\alpha,s))^{\frac{1}{2}}\frac{dy}{d\mu} \tag{4.2}$$

If we take the standard derivative of (4.2) with s, then

$$\frac{d}{ds}\left(\frac{d^\alpha y}{ds^\alpha}\right) = \frac{d}{ds}(C(\alpha,s))^{\frac{1}{2}}\frac{dy}{d\mu} + (C(\alpha,s))^{\frac{1}{2}}\frac{d^2y}{d\mu^2}\frac{d\mu}{ds}$$

or

$$\frac{d}{ds}\left(\frac{d^\alpha y}{ds^\alpha}\right) = \frac{d}{ds}(C(\alpha,s))^{\frac{1}{2}}\frac{dy}{d\mu} + \frac{d^2y}{d\mu^2} \tag{4.3}$$

If we take the standard derivative of (4.3) with s, then

$$\frac{d^2}{ds^2}\left(\frac{d^\alpha y}{ds^\alpha}\right) = \frac{d^2}{ds^2}(C(\alpha,s))^{\frac{1}{2}}\frac{dy}{d\mu} + \frac{d}{ds}(C(\alpha,s))^{\frac{1}{2}}\frac{d^2y}{d\mu^2}\frac{d\mu}{ds} + \frac{d^3y}{d\mu^3}\frac{d\mu}{ds}$$

or

$$\frac{d^2}{ds^2}\left(\frac{d^\alpha y}{ds^\alpha}\right) = \frac{d^2}{ds^2}(C(\alpha,s))^{\frac{1}{2}}\frac{dy}{d\mu} + \frac{d}{ds}(C(\alpha,s))^{\frac{1}{2}}(C(\alpha,s))^{-\frac{1}{2}}\frac{d^2y}{d\mu^2} + (C(\alpha,s))^{-\frac{1}{2}}\frac{d^3y}{d\mu^3} \tag{4.4}$$

Thus,

$$\left[\frac{d^\alpha y}{ds^\alpha}\ \ \frac{d}{ds}\left(\frac{d^\alpha y}{ds^\alpha}\right)\ \ \frac{d^2}{ds^2}\left(\frac{d^\alpha y}{ds^\alpha}\right)\right] = (C(\alpha,s))^{\frac{1}{2}}(C(\alpha,s))^{-\frac{1}{2}}\left[\frac{dy}{d\mu}\ \ \frac{d^2y}{d\mu^2}\ \ \frac{d^3y}{d\mu^3}\right] = 1$$

□

Let y(s), $s \in (c,d), 0 < c < d$, be a parametrized curve in $\mathbb{R}^3$ with α-equiaffine arc length. Then, the set $\left\{e_1^{\{\alpha\}}, e_2^{\{\alpha\}}, e_3^{\{\alpha\}}\right\}$ is called α-equiaffine Frenet frame of $y(s)$, where $e_1^{\{\alpha\}} = \frac{d^\alpha y}{ds^\alpha}$, $e_2^{\{\alpha\}} = \frac{d}{ds}\left(\frac{d^\alpha y}{ds^\alpha}\right)$, and $e_3^{\{\alpha\}} = \frac{d^2}{ds^2}\left(\frac{d^\alpha y}{ds^\alpha}\right)$. Note that when $\alpha = 1$ the set $\left\{e_1^{\{\alpha\}}, e_2^{\{\alpha\}}, e_3^{\{\alpha\}}\right\}$ is the standard equiaffine Frenet frame of $y(s)$. Therefore,

$$\left[e_1^{\{\alpha\}}\ e_2^{\{\alpha\}}\ e_3^{\{\alpha\}}\right] = 1$$

and if we take the standard derivative of the last equation according to s, then

$$\left[e_1^{\{\alpha\}} \;\; e_2^{\{\alpha\}} \;\; \frac{d}{ds}\left(e_3^{\{\alpha\}}\right)\right] = 0$$

where it can be seen that the set $\left\{e_1^{\{\alpha\}}, e_2^{\{\alpha\}}, \frac{d}{ds}\left(e_3^{\{\alpha\}}\right)\right\}$ is linearly dependent for every $s \in (c, d)$. Then, there are some smooth functions on (c, d) denoted by $\kappa^{\{\alpha\}}(s)$ and $\tau^{\{\alpha\}}(s)$ such that

$$\kappa^{\{\alpha\}}(s)\, e_1^{\{\alpha\}} + \tau^{\{\alpha\}}(s)\, e_2^{\{\alpha\}} + \frac{d}{ds}\left(e_3^{\{\alpha\}}\right) = 0$$

Hence, the functions $\kappa^{\{\alpha\}}(s)$ and $\tau^{\{\alpha\}}(s)$ are defined *α-equiaffine curvatures* of $y(s)$, where

$$\kappa^{\{\alpha\}}(s) = -\left[e_2^{\{\alpha\}} \;\; e_3^{\{\alpha\}} \;\; \frac{d}{ds}\left(e_3^{\{\alpha\}}\right)\right] \tag{4.5}$$

and

$$\tau^{\{\alpha\}}(s) = \left[e_1^{\{\alpha\}} \;\; e_3^{\{\alpha\}} \;\; \frac{d}{ds}\left(e_3^{\{\alpha\}}\right)\right] \tag{4.6}$$

According to the above discussion, we have the counterpart of equiaffine Frenet formulas as follows:

$$\begin{bmatrix} \frac{d}{ds}\left(e_1^{\{\alpha\}}\right) \\ \frac{d}{ds}\left(e_2^{\{\alpha\}}\right) \\ \frac{d}{ds}\left(e_3^{\{\alpha\}}\right) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\kappa^{\{\alpha\}} & -\tau^{\{\alpha\}} & 0 \end{bmatrix} \begin{bmatrix} e_1^{\{\alpha\}} \\ e_2^{\{\alpha\}} \\ e_3^{\{\alpha\}} \end{bmatrix}$$

We may present the relationship between the equiaffine frames of $y(s)$. For this, let $\mathcal{B}^{\{\alpha\}} = \left[e_1^{\{\alpha\}} \; e_2^{\{\alpha\}} \; e_3^{\{\alpha\}}\right]$ and $\mathcal{B} = [e_1 \; e_2 \; e_3]$, for $\mathcal{B}^{\{\alpha\}}, \mathcal{B} \in SL(3, \mathbb{R})$. Thus, using (4.2), (4.3), and (4.4),

$$\mathcal{B}^{\{\alpha\}} = \begin{bmatrix} (C(\alpha,s))^{\frac{1}{2}} & 0 & 0 \\ \frac{d}{ds}(C(\alpha,s))^{\frac{1}{2}} & 1 & 0 \\ \frac{d^2}{ds^2}(C(\alpha,s))^{\frac{1}{2}} & (C(\alpha,s))^{-\frac{1}{2}} \frac{d}{ds}(C(\alpha,s))^{\frac{1}{2}} & (C(\alpha,s))^{-\frac{1}{2}} \end{bmatrix} \mathcal{B}$$

Theorem 4.2. Let $y(s)$ be a non-degenerate smooth curve in $\mathbb{R}^3$ parameterized by equiaffine arc length parameter concerning (2.3). Then,

$$\begin{aligned} \kappa^{\{\alpha\}}(s) = &-\frac{3}{4}(C(\alpha,s))^{-3}\left(\frac{d}{ds}(C(\alpha,s))\right)^3 \\ &+ \frac{5}{4}(C(\alpha,s))^{-2}\frac{d}{ds}(C(\alpha,s))\frac{d^2}{ds^2}(C(\alpha,s)) - \frac{1}{2}(C(\alpha,s))^{-1}\frac{d^3}{ds^3}(C(\alpha,s)) \\ &+ (C(\alpha,s))^{-\frac{3}{2}}\kappa(\mu(s)) - \frac{1}{2}(C(\alpha,s))^{-2}\frac{d}{ds}(C(\alpha,s))\,\tau(\mu(s)) \end{aligned} \tag{4.7}$$

and

$$\tau^{\{\alpha\}}(s) = \frac{3}{4}(C(\alpha,s))^{-2}\left(\frac{d}{ds}(C(\alpha,s))\right)^2 - (C(\alpha,s))^{-1}\frac{d^2}{ds^2}(C(\alpha,s)) + (C(\alpha,s))^{-1})\tau(\mu(s)) \tag{4.8}$$

Here, κ, τ and $\kappa^{\{\alpha\}}$, $\tau^{\{\alpha\}}$ denote the equiaffine and α-equiaffine curvatures, respectively.

Proof. Differentiating (4.4) with respect to s, we have

$$\frac{d}{ds}e_3^{\{\alpha\}} = \phi(s)e_1^{\{\alpha\}} + \zeta(s)e_2^{\{\alpha\}} \tag{4.9}$$

where

$$\phi(s) = \frac{d^3}{ds^3}(C(\alpha,s))^{\frac{1}{2}} - (C(\alpha,s))^{-1}\kappa(\mu(s))$$

and

$$\zeta(s) = 2\frac{d^2}{ds^2}\left(C(\alpha,s)\right)^{\frac{1}{2}}\left(C(\alpha,s)\right)^{-\frac{1}{2}} + \frac{d}{ds}\left(C(\alpha,s)\right)^{\frac{1}{2}}\frac{d}{ds}\left(C(\alpha,s)\right)^{-\frac{1}{2}} - \left(C(\alpha,s)\right)^{-1}\tau(\mu(s))$$

If we consider (4.3), (4.4), and (4.9) in (4.5), after some calculations, we get (4.7). Analogously, if we consider (4.2), (4.4), and (4.9) in (4.6), we get (4.8). □

Remark 4.3. In particular if $C(\alpha,s) = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}$ as in [24], then Theorem 4.2 reduces to [24, Theorem 4.1].

Remark 4.4. If the conformable derivative is taken into consideration as our derivative, since $C(\alpha,s) = s^{1-\alpha}$, Theorem 4.2 can be given as follows:

$$\kappa^{\{\alpha\}}(s) = \frac{(\alpha+3)(\alpha-1)}{4}s^{-3} + s^{\frac{3\alpha-3}{2}}\kappa(\mu(s)) - \frac{(1-\alpha)}{2}s^{\alpha-2}\tau(\mu(s)) \tag{4.10}$$

and

$$\tau^{\{\alpha\}}(s) = \frac{(\alpha+3)(1-\alpha)}{4}s^{-2} + s^{\alpha-1}\tau(\mu(s)) \tag{4.11}$$

Suppose that $C(\alpha,s) = s^{1-\alpha}$, namely our derivative is conformable type. Let $\kappa(\mu(s)) = -2, \tau(\mu(s)) = -3$ (respectively, $\kappa(\mu(s)) = 2, \tau(\mu(s)) = 3$ and $\kappa(\mu(s)) = 0, \tau(\mu(s)) = 0$). Then, the graphs of (4.10) are as in Figure 1. Similarly, the graphs of (4.11) are as in Figure 2.

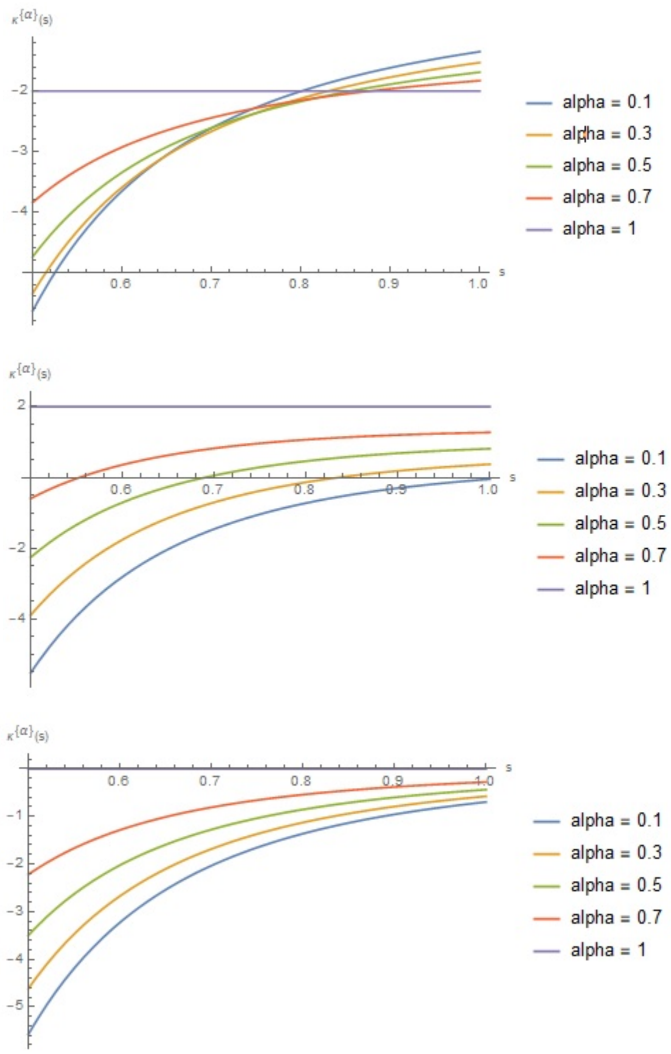

Figure 1. Graphs of $\kappa^{\{\alpha\}}(s)$ for $\kappa(\mu(s)) = -2, \tau(\mu(s)) = -3$ (respectively, $\kappa(\mu(s)) = 2, \tau(\mu(s)) = 3$ and $\kappa(\mu(s)) = 0, \tau(\mu(s)) = 0$) and $s \in [0.5, 1]$

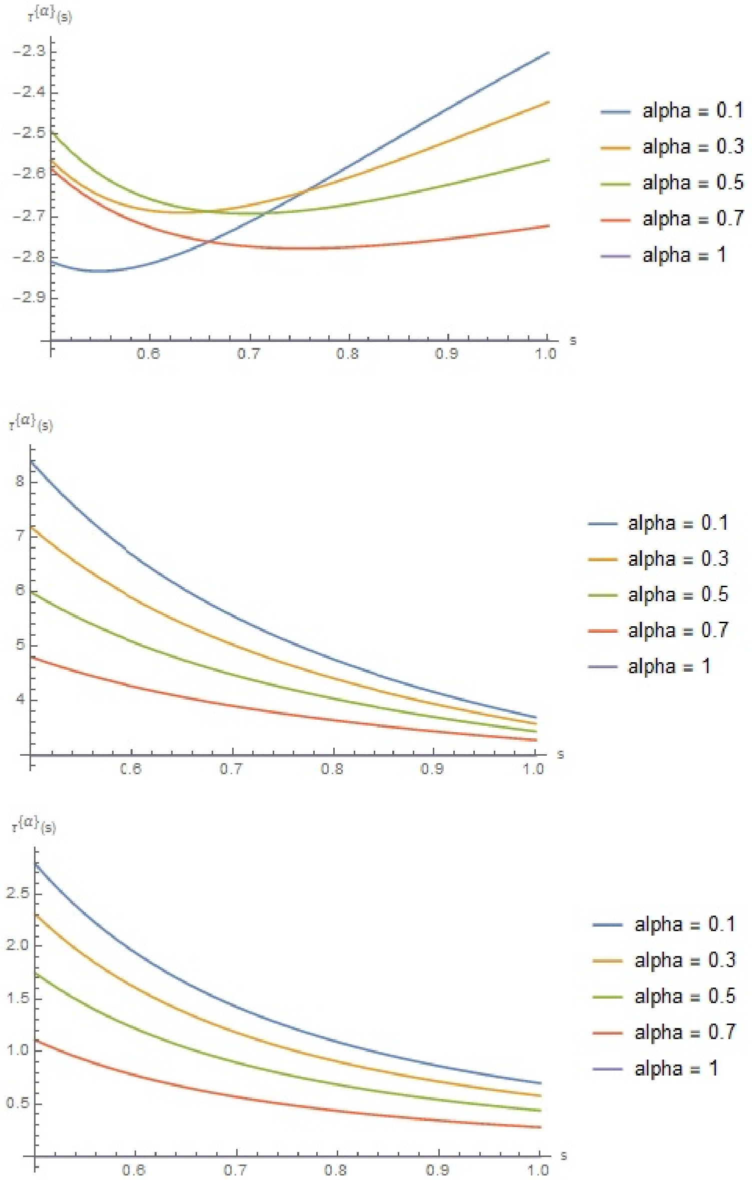

Figure 2. Graphs of $\tau^{\{\alpha\}}(s)$ for $\tau(\mu(s)) = -3$ (respectively, $\tau(\mu(s)) = 3, \tau(\mu(s)) = 0$) and $s \in [0.5, 1]$

Example 4.5. Consider conformable derivative as the derivative and the following curve $y(\mu)$ in $\mathbb{R}^3$ (see Figure 3)

$$y(\mu) = (\cos\mu, \sin\mu, \mu), \mu \in (a, b), 0 < a < b$$

where μ is the equiaffine arc length parameter of $y(\mu)$. Considering (3.2) and (3.3), $\kappa(\mu) = 0$ and $\tau(\mu) = 1$, for $y(\mu)$. Suppose that $C(\alpha, s) = s^{1-\alpha}$, namely our derivative is conformable type. Then, $\kappa^{\{\alpha\}}(s)$ and $\tau^{\{\alpha\}}(s)$ are obtained as follows:

$$\kappa^{\{\alpha\}}(s) = \frac{(\alpha+3)(\alpha-1)}{4}s^{-3} - \frac{1-\alpha}{2}s^{\alpha-2}$$

and

$$\tau^{\{\alpha\}}(s) = \frac{(\alpha+3)(1-\alpha)}{4}s^{-2} + s^{\alpha-1}$$

The graphs of $\kappa^{\{\alpha\}}(s)$ and $\tau^{\{\alpha\}}(s)$ for $y(\mu) = (\cos\mu, \sin\mu, \mu)$ are as in Figure 4.

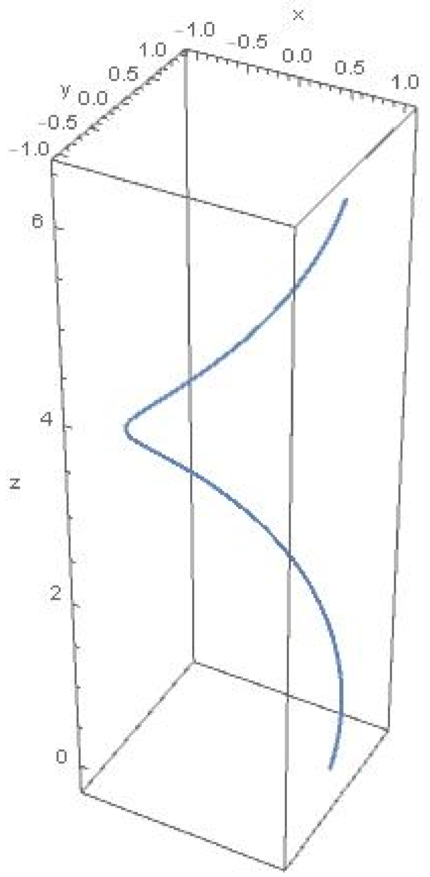

Figure 3. Graphs of the $y(\mu) = (\cos\mu, \sin\mu, \mu)$ function) and $s \in [0.5, 1]$

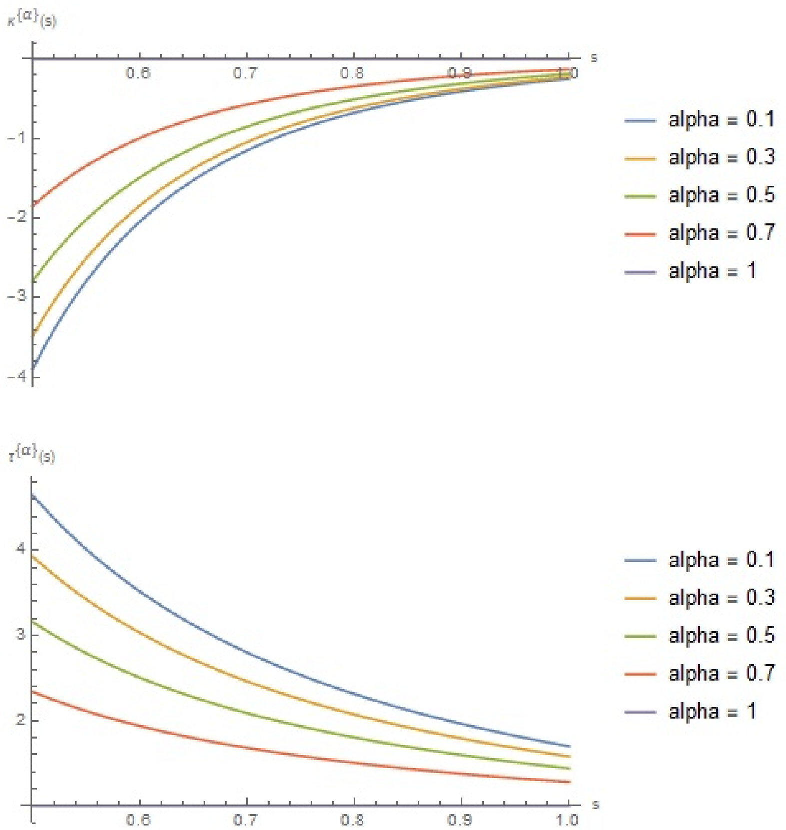

Figure 4. Graphs of α-equiaffine curvatures of $y(\mu) = (\cos\mu, \sin\mu, \mu)$ for $s \in [0.5, 1]$

Corollary 4.6. Let $y(s)$, $s \in (c,d), 0 < c < d$, be a parametrized curve in $\mathbb{R}^3$ with α-equiaffine arc length. If the equiaffine curvatures of $y(s)$ vanish identically, then

$$\begin{aligned}\kappa^{\{\alpha\}}(s) &= -\tfrac{3}{4}\left(C(\alpha,s)\right)^{-3}\left(\tfrac{d}{ds}\left(C(\alpha,s)\right)\right)^3 \tfrac{5}{4}\left(C(\alpha,s)\right)^{-2}\tfrac{d}{ds}\left(C(\alpha,s)\right)\tfrac{d^2}{ds^2}\left(C(\alpha,s)\right)\\ &\quad -\tfrac{1}{2}\left(C(\alpha,s)\right)^{-1}\tfrac{d^3}{ds^3}\left(C(\alpha,s)\right)\end{aligned}$$

and

$$\tau^{\{\alpha\}}(s) = \frac{3}{4}\left(C(\alpha,s)\right)^{-2}\left(\frac{d}{ds}\left(C(\alpha,s)\right)\right)^2 - \left(C(\alpha,s)\right)^{-1}\frac{d^2}{ds^2}\left(C(\alpha,s)\right)$$

Proof. It follows by (4.7) and (4.8). □

Corollary 4.7. Let $y(\mu)$, $\mu \in (c,d), 0 < c < d$, be a parametrized curve in $\mathbb{R}^3$ with α-equiaffine arc length. Then,

$$\begin{aligned}\kappa(\mu) &= \tfrac{1}{4}\left(C(\alpha,s)\right)^{-\frac{3}{2}}\left(\tfrac{d}{ds}\left(C(\alpha,s)\right)\right)^3 - \tfrac{3}{4}\left(C(\alpha,s)\right)^{-\frac{1}{2}}\tfrac{d}{ds}\left(C(\alpha,s)\right)\tfrac{d^2}{ds^2}\left(C(\alpha,s)\right)\\ &\quad +\tfrac{1}{2}\left(C(\alpha,s)\right)^{\frac{1}{2}}\tfrac{d^2}{ds^2}\left(C(\alpha,s)\right) + \tfrac{1}{2}\left(C(\alpha,s)\right)^{\frac{1}{2}}\tfrac{d}{ds}\left(C(\alpha,s)\right)\tau^{\{\alpha\}}(s) + \left(C(\alpha,s)\right)^{\frac{3}{2}}\kappa^{\{\alpha\}}(s)\end{aligned} \tag{4.12}$$

and

$$\tau(\mu) = \frac{d^2}{ds^2}\left(C(\alpha,s)\right) - \frac{3}{4}\left(C(\alpha,s)\right)^{-1}\left(\frac{d}{ds}\left(C(\alpha,s)\right)\right)^2 + \left(C(\alpha,s)\right)\tau^{\{\alpha\}}(s) \tag{4.13}$$

Proof. It follows by (4.7) and (4.8). □

Corollary 4.8. Let $y(\mu)$, $\mu \in (c,d), 0 < c < d$, be a parametrized curve in $\mathbb{R}^3$ with α-equiaffine arc length. If $\kappa^{\{\alpha\}}(s) = 0$ and $\tau^{\{\alpha\}}(s) = 0$, then

$$\begin{aligned}\kappa(\mu) &= \tfrac{1}{4}\left(C(\alpha,s)\right)^{-\frac{3}{2}}\left(\tfrac{d}{ds}\left(C(\alpha,s)\right)\right)^3 - \tfrac{3}{4}\left(C(\alpha,s)\right)^{-\frac{1}{2}}\tfrac{d}{ds}\left(C(\alpha,s)\right)\tfrac{d^2}{ds^2}\left(C(\alpha,s)\right)\\ &\quad +\tfrac{1}{2}\left(C(\alpha,s)\right)^{\frac{1}{2}}\tfrac{d^2}{ds^2}\left(C(\alpha,s)\right)\end{aligned}$$

and

$$\tau(\mu) = \frac{d^2}{ds^2}\left(C(\alpha,s)\right) - \frac{3}{4}\left(C(\alpha,s)\right)^{-1}\left(\frac{d}{ds}\left(C(\alpha,s)\right)\right)^2$$

Proof. It is obvious from (4.12) and (4.13). □

5. Conclusion

By incorporating a general local fractional derivative, this paper enhances the theory of equiaffine curves in the 3-dimensional affine space $\mathbb{R}^3$. We introduce novel invariants for these curves and establish connections between these new invariants and the conventional ones. Furthermore, we derive new results for equiaffine plane curves belonging to $\kappa^{\{\alpha\}}$ and $\tau^{\{\alpha\}}$. These bring forth a different perspective in affine geometry, leveraging the characteristics of fractional calculus. Additionally, Figures 1 and 2 show graphs indicating the relationship between specially selected equiaffine curvatures and α-equiafine curvatures of a curve. Moreover, Figures 3 and 4 show a provided curve in Example 4.5 and the graph of α-equiafine curvatures of this curve. These graphs show the behavior of α-equiafine curvatures for different α-values. It is clear that if the approach and calculations discussed in this study are considered in studies to be carried out in fractional derivative and affine space, a more general version of the characterizations obtained will be obtained. Therefore, this situation is an open problem, especially for researchers who will study invariants of curves. Hence, we can pose the following problem: To find the relations in higher dimensions between the fractional and standard equiaffine curvatures, akin to (4.7) and (4.8). Specifically, the primary objective of this problem is to formulate one equation expressing the relations between the fractional and standard equiaffine curvatures.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

References

[1] D. Baleanu, A. Fernandez, *On fractional operators and their classifications*, Mathematics 7 (9) (2019) 830 10 pages.

[2] M. E. Aydin, A. Mihai, A. Yokus, *Applications of fractional calculus in equiaffine geometry: Plane curves with fractional order*, Mathematical Methods in the Applied Sciences 44 (17) (2020) 13659–13669.

[3] T. Yajima, S. Oiwa, K. Yamasaki, *Geometry of curves with fractional-order tangent vector and Frenet-Serret formulas*, Fractional Calculus and Applied Analysis 21 (6) (2018) 1493–1505.

[4] M. E. Aydın, M. Bektaş, A. O. Öğrenmis, A. Yokuş, *Differential geometry of curves in Euclidean 3-space with fractional order*, International Electronic Journal of Geometry 14 (1) (2021) 132–144.

[5] U. Gözütok, H. A. Çoban, Y. Sağıroğlu, *Frenet frame with respect to conformable derivative*, Filomat 33 (6) (2019) 1541–1550.

[6] A. Has, B. Yılmaz, *Special fractional curve pairs with fractional calculus*, International Electronic Journal of Geometry 15 (1) (2022) 132–144.

[7] K. Lazopoulos, A. K. Lazopoulos, *Fractional differential geometry of curves and surfaces*, Progress in Fractional Differentiation and Applications 2 (3) (2016) 169–186.

[8] V. E. Tarasov, *On chain rule for fractional derivatives*, Communications in Nonlinear Science and Numerical Simulation 30 (1) (2016) 1–4.

[9] H. Bulut, H. M. Baskonus, Y. Pandir, *The modified trial equation method for fractional wave equation and time fractional generalized burgers equation*, Abstract and Applied Analysis 2013 (2013) 636802 8 pages.

[10] H. F. Ismael, H. M. Baskonus, H. Bulut, W. Gao, *Instability modulation and novel optical soliton solutions to the Gerdjikov–Ivanov equation with m-fractional*, Optical and Quantum Electronics 55 (4) (2023) 303.

[11] M. A. Dokuyucu, E. Çelik, H. Bulut, H. M. Baskonus, *Cancer treatment model with the Caputo-Fabrizio fractional derivative*, The European Physical Journal Plus 133 (3) (2018) 92.

[12] R. L. Magin, Fractional Calculus in Bioengineering, Begell House, Connecticut, Chicago, 2006.

[13] T. Yajima, H. Nagahama, *Differential geometry of viscoelastic models with fractional-order derivatives*, Journal of Physics A: Mathematical and Theoretical 43 (38) (2010) 385207 9 pages.

[14] D. Baleanu, S. S. Sajjadi, A. Jajarmi, Ö. Defterli, *On a nonlinear dynamical system with both chaotic and nonchaotic behaviors: A new fractional analysis and control*, Advances in Difference Equations 2021 (2021) 234 17 pages.

[15] T. Yajima, H. Nagahama, *Geometric structures of fractional dynamical systems in non-Riemannian space: Applications to mechanical and electromechanical systems*, Annalen der Physik 530 (5) (2018) 1700391 9 pages.

[16] I. Podlubny, *Fractional Differential Equations, Mathematics in Science and Engineering*, Academic Press, New York, 1999.

[17] T. Abdeljawad, *On conformable fractional calculus*, Journal of Computational and Applied Mathematics 279 (2015) 57–66.

[18] J. Vanterler da C. Sousa, E. Capelas de Oliveira, *A new truncated m-fractional derivative type unifying some fractional derivative types with classical properties*, International Journal of Analysis and Applications 16 (1) (2018) 83–96.

[19] J. Vanterler da C. Sousa, E. Capelas de Oliveira, *Mittag–Leffler functions and the truncated ν-fractional derivative*, Mediterranean Journal of Mathematics 14 (6) (2017) 244 24 pages.

[20] A. Kilbas, H. Srivastava, J. Trujillo, Theory and applications of fractional differential equations, North-Holland Mathematics Studies, New York, 2006.

[21] J. N. Clelland, From Frenet to Cartan: The method of moving frames, American Mathematical Society, Providence, 2017.

[22] D. Davis, *Generic affine differential geometry of curves in* $\mathbb{R}^n$, Proceedings of the Royal Society of Edinburgh Section A 136 (6) (2006) 1195–1205.

[23] H. W. Guggenheimer, Differential geometry, McGraw-Hill, New York, 1963.

[24] M. E. Aydın, S. Kaya, *Fractional equiaffine curvatures of curves in 3-dimensional affine space*, International Journal of Maps in Mathematics 6 (1) (2023) 67–82.