Adıyaman University Journal of Science
https://dergipark.org.tr/en/pub/adyujsci
DergiPark AKADEMİK
ISSN 2147-1630
e-ISSN 2146-586X

Invariant and Lacunary Invariant Statistical Equivalence of Order β for Double Set Sequences

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Received: 24.06.2021 **Accepted:** 24.11.2021 **Published:** 31.12.2021

Abstract

In this study, as a new approach to the concept of asymptotical equivalence in the Wijsman sense for double set sequences, the new concepts which are called asymptotical invariant statistical equivalence of order β and asymptotical lacunary invariant statistical equivalence of order β $(0 < \beta \leq 1)$ in the Wijsman sense for double set sequences are introduced and explained with examples. In addition, the existence of some relations between these concepts and furthermore, the relationships between these concepts and previously studied asymptotical equivalence concepts in the Wijsman sense for double set sequences are investigated.

Keywords: Asymptotical equivalence; Convergence in the Wijsman sense; Double lacunary sequence; Invariant statistical convergence; Order β; Sequences of sets.

Çift Küme Dizileri için β ıncı Mertebeden İnvaryant ve Lacunary İnvaryant İstatistiksel Denklik

* Corresponding Author DOI: 10.37094/adyujsci.956986

Öz

Bu çalışmada, çift küme dizileri için Wijsman anlamında asimptotik denklik kavramına yeni bir yaklaşım olarak, çift küme dizileri için Wijsman anlamında β $(0 < \beta \leq 1)$ yıncı mertebeden asimptotik invaryant istatistiksel denklik ve asimptotik lacunary invaryant istatistiksel denklik olarak adlandırılan yeni kavramlar tanıtıldı ve örneklerle açıklandı. Ayrıca, bu kavramlar arasında bazı ilişkilerin varlığı ve dahası bu kavramlar ve daha önceden çift küme dizileri için Wijsman anlamında çalışılmış asimptotik denklik kavramları arasındaki ilişkiler incelendi.

Anahtar Kelimeler: Asimptotik denklik; Wijsman anlamında yakınsaklık; Çift lacunary dizi; İnvaryant istatistiksel yakınsaklık; β ıncı mertebe; Küme dizisi.

1. Introduction

Long after the concept of convergence for double sequences was introduced by Pringsheim [1], using the concepts of statistical convergence, double lacunary sequence and σ-convergence, this concept was extended to new convergence concepts for double sequences by some authors [2-4]. Recently, for double sequences, on two new convergence concepts called double almost statistical and double almost lacunary statistical convergence of order α were studied by Savaş [5, 6]. Moreover, for double sequences, the concept of asymptotical equivalence was introduced by Patterson [7].

Over the years, on the various convergence concepts for set sequences were studied by several authors. One of them, discussed in this study, is the concept of convergence in the Wijsman sense [8, 9]. Using the concepts of statistical convergence, double lacunary sequence and σ-convergence, this concept was extended to new convergence concepts for double set sequences by some authors [10-12]. In [11], Nuray and Ulusu studied on the concepts of invariant statistical and lacunary invariant statistical convergence in the Wijsman sense for double set sequences. Furthermore, for double set sequences, the concepts of asymptotical equivalence in the Wijsman sense were introduced by Nuray et al. [13] and then these concepts were studied by some authors.

In this paper, using order β, we studied on new asymptotical equivalence concepts in the Wijsman sense for double set sequences.

More information on the concepts of convergence or asymptotical equivalence for real or set sequences can be found in [14-24].

2. Preliminaries

First of all, let us recall the basic notions necessary for a better understanding of our study [3, 10, 12, 13, 25].

For a metric space (Y, d), $\rho(y, C)$ denote the distance from y to C where

$$\rho(y, C) := \rho_y(C) = \inf_{c \in C} d(y, c),$$

for any $y \in Y$ and any non empty set $C \subseteq Y$.

For a non empty set Y, let a function $g: \mathbb{N} \to 2^Y$ (the power set of Y) is defined by $g(m) = C_m \in 2^Y$ for each $m \in \mathbb{N}$. Then the sequence $\{C_m\} = \{C_1, C_2, \dots\}$, which is the codomain elements of g, is called set sequences.

Throughout this study, (Y, d) will be considered as a metric space and C, C_{mn}, D_{mn} will be considered as any non empty closed subsets of Y.

A double set sequence $\{C_{mn}\}$ is called convergent to the set C in the Wijsman sense if each $y \in Y$,

$$\lim_{m,n \to \infty} \rho_y(C_{mn}) = \rho_y(C).$$

A double set sequence $\{C_{mn}\}$ is called statistical convergent to the set C in the Wijsman sense if every $\xi > 0$ and each $y \in Y$,

$$\lim_{p,q \to \infty} \frac{1}{pq} \left| \left\{ (m,n) : m \leq p, n \leq q, |\rho_y(C_{mn}) - \rho_y(C)| \geq \xi \right\} \right| = 0.$$

A double sequence $\theta_2 = \{(j_s, k_t)\}$ is called a double lacunary sequence if there exist increasing sequences (j_s) and (k_t) of the integers such that

$$j_0 = 0,\ h_s = j_s - j_{s-1} \to \infty \text{ and } k_0 = 0,\ \bar{h}_t = k_t - k_{t-1} \to \infty \text{ as } s, t \to \infty.$$

In general, the following notations is used for any double lacunary sequence:

$$\ell_{st} = j_s k_t,\ h_{st} = h_s \bar{h}_t,\ I_{st} = \{(m,n) : j_{s-1} < m \leq j_s \text{ and } k_{t-1} < n \leq k_t\},$$

$$q_s = \frac{j_s}{j_{s-1}} \text{ and } q_t = \frac{k_t}{k_{t-1}}.$$

Throughout this study, $\theta_2 = \{(j_s, k_t)\}$ will be considered as a double lacunary sequence.

A double set sequence $\{C_{mn}\}$ is called lacunary statistical convergent to the set C in the Wijsman sense if every $\xi > 0$ and each $y \in Y$,

$$\lim_{s,t\to\infty}\frac{1}{h_{st}}\left|\{(m,n)\in I_{st}: |\rho_y(C_{mn})-\rho_y(C)|\geq\xi\}\right|=0.$$

The term $\rho_y\left(\frac{C_{mn}}{D_{mn}}\right)$ is defined as follows:

$$\rho_y\left(\frac{C_{mn}}{D_{mn}}\right)=\begin{cases}\dfrac{\rho(y,C_{mn})}{\rho(y,D_{mn})} & , \quad y\notin C_{mn}\cup D_{mn}\\ \lambda & , \quad y\in C_{mn}\cup D_{mn}.\end{cases}$$

Double set sequences $\{C_{mn}\}$ and $\{D_{mn}\}$ are called asymptotically equivalent in the Wijsman sense if each $y\in Y$,

$$\lim_{m,n\to\infty}\rho_y\left(\frac{C_{mn}}{D_{mn}}\right)=1$$

and denoted by $C_{mn}\overset{W}{\sim}D_{mn}$.

Let σ be a mapping such that $\sigma:\mathbb{N}^+\to\mathbb{N}^+$ (the set of positive integers). A continuous linear functional ψ on ℓ_∞ is called an invariant mean (or a σ-mean) if it satisfies the following conditions:

1. $\psi(x_u)\geq 0$, when the sequence (x_u) has $x_u\geq 0$ for all u,

2. $\psi(e)=1$, where $e=(1,1,1,\ldots)$ and

3. $\psi(x_{\sigma(u)})=\psi(x_u)$ for all $(x_u)\in\ell_\infty$.

The mappings σ are assumed to be one to one and such that $\sigma^m(u)\neq u$ for all $m,u\in\mathbb{N}^+$, where $\sigma^m(u)$ denotes the m th iterate of the mapping σ at u. Thus ψ extends the limit functional on c, in the sense that $\psi(x_u)=\lim x_u$ for all $(x_u)\in c$.

Double set sequences $\{C_{mn}\}$ and $\{D_{mn}\}$ are called asymptotically invariant statistical equivalent to multiple λ in the Wijsman sense if every $\xi>0$ and each $y\in Y$,

$$\lim_{p,q\to\infty}\frac{1}{pq}\left|\left\{(m,n): m\leq p, n\leq q, \left|\rho_y\left(\frac{C_{\sigma^m(u)\sigma^n(v)}}{D_{\sigma^m(u)\sigma^n(v)}}\right)-\lambda\right|\geq\xi\right\}\right|=0$$

uniformly in u,v.

The set of all asymptotically invariant statistical equivalent to multiple λ double set sequences in the Wijsman sense is denoted by $\{W_2^\lambda(S_\sigma)\}$.

Double set sequences $\{C_{mn}\}$ and $\{D_{mn}\}$ are called asymptotically lacunary invariant statistical equivalent to multiple λ in the Wijsman sense if every $\xi>0$ and each $y\in Y$,

$$\lim_{s,t\to\infty} \frac{1}{h_{st}} \left| \left\{ (m,n) \in I_{st}: \left| \rho_y \left(\frac{C_{\sigma^m(u)\sigma^n(v)}}{D_{\sigma^m(u)\sigma^n(v)}} \right) - \lambda \right| \geq \xi \right\} \right| = 0$$

uniformly in u, v.

3. Main Results

In this section, for double set sequences, the concepts of asymptotical invariant statistical and asymptotical lacunary invariant statistical equivalence of order β $(0 < \beta \leq 1)$ in the Wijsman sense were introduced. Also, the inclusion relations between them were investigated.

Definition 1. Double set sequences $\{C_{mn}\}$ and $\{D_{mn}\}$ are asymptotically invariant statistical equivalent to multiple λ of order β in the Wijsman sense if every $\xi > 0$ and each $y \in Y$,

$$\lim_{p,q\to\infty} \frac{1}{(pq)^\beta} \left| \left\{ (m,n): m \leq p, n \leq q, \left| \rho_y \left(\frac{C_{\sigma^m(u)\sigma^n(v)}}{D_{\sigma^m(u)\sigma^n(v)}} \right) - \lambda \right| \geq \xi \right\} \right| = 0$$

uniformly in u, v, where $0 < \beta \leq 1$ and we denote this in $C_{mn} \overset{W_2^\lambda(S_\sigma^\beta)}{\sim} D_{mn}$ format, and simply called asymptotically invariant statistical equivalent of order β in the Wijsman sense if $\lambda = 1$.

Example 1. Let $Y = \mathbb{R}^2$ and double set sequences $\{C_{mn}\}$ and $\{D_{mn}\}$ be defined as following:

$$C_{mn} := \begin{cases} \left\{ (a,b) \in \mathbb{R}^2 : a^2 + (b-1)^2 = \frac{1}{mn} \right\} & ; \quad \text{if } m \text{ and } n \text{ are square integers} \\ \{(1,0)\} & ; \quad \text{otherwise.} \end{cases}$$

and

$$D_{mn} := \begin{cases} \left\{ (a,b) \in \mathbb{R}^2 : a^2 + (b+1)^2 = \frac{1}{mn} \right\} & ; \quad \text{if } m \text{ and } n \text{ are square integers} \\ \{(1,0)\} & ; \quad \text{otherwise.} \end{cases}$$

In this case, the double set sequences $\{C_{mn}\}$ and $\{D_{mn}\}$ are asymptotically invariant statistical equivalent of order β $(0 < \beta \leq 1)$ in the Wijsman sense.

Remark 1. For $\beta = 1$, the concept of asymptotical invariant statistical equivalence of order β in the Wijsman sense coincides with the concept of asymptotical invariant statistical equivalence in the Wijsman sense for double set sequences in [25].

Definition 2. Double set sequences $\{C_{mn}\}$ and $\{D_{mn}\}$ are asymptotically lacunary invariant statistical equivalent to multiple λ of order β in the Wijsman sense if every $\xi > 0$ and each $y \in Y$,

$$\lim_{s,t\to\infty} \frac{1}{h_{st}^{\beta}} \left| \left\{ (m,n) \in I_{st} : \left| \rho_y \left(\frac{C_{\sigma^m(u)\sigma^n(v)}}{D_{\sigma^m(u)\sigma^n(v)}} \right) - \lambda \right| \geq \xi \right\} \right| = 0$$

uniformly in u, v, where $0 < \beta \leq 1$ and we denote this in $C_{mn} \overset{W_2^{\lambda}(S_{\sigma\theta}^{\beta})}{\sim} D_{mn}$ format, and simply called asymptotical lacunary invariant statistical equivalent of order β in the Wijsman sense if $\lambda = 1$.

The set of all asymptotically lacunary invariant statistically equivalent double set sequences to multiple λ of order β in the Wijsman sense is denoted by $\{W_2^{\lambda}(S_{\sigma\theta}^{\beta})\}$.

Example 2. Let $Y = \mathbb{R}^2$ and double set sequences $\{C_{mn}\}$ and $\{D_{mn}\}$ be defined as following:

$$C_{mn} := \begin{cases} \{(a,b) \in \mathbb{R}^2 : (a-m)^2 + (b+n)^2 = 1\} & ; \quad \text{if } (m,n) \in I_{st}, \\ & \quad m \text{ and } n \text{ are square integer} \\ \{(-1,-1)\} & ; \quad \text{otherwise.} \end{cases}$$

and

$$D_{mn} := \begin{cases} \{(a,b) \in \mathbb{R}^2 : (a+m)^2 + (b-n)^2 = 1\} & ; \quad \text{if } (m,n) \in I_{st}, \\ & \quad m \text{ and } n \text{ are square integer} \\ \{(-1,-1)\} & ; \quad \text{otherwise.} \end{cases}$$

In this case, the double set sequences $\{C_{mn}\}$ and $\{D_{mn}\}$ are asymptotically lacunary invariant statistical equivalent of order β $(0 < \beta \leq 1)$ in the Wijsman sense.

Remark 2. For $\beta = 1$, the concept of asymptotical lacunary invariant statistical equivalence of order β in the Wijsman sense coincide with the concept of asymptotical lacunary invariant statistical equivalence in the Wijsman sense for double set sequences in [25].

Theorem 1. If $\liminf_s q_s^{\beta} > 1$ and $\liminf_t q_t^{\beta} > 1$ where $0 < \beta \leq 1$, then

$$C_{mn} \overset{W_2^{\lambda}(S_{\sigma}^{\beta})}{\sim} D_{mn} \Rightarrow C_{mn} \overset{W_2^{\lambda}(S_{\sigma\theta}^{\beta})}{\sim} D_{mn}.$$

Proof. Let $0 < \beta \leq 1$ and suppose that $\liminf_s q_s^{\beta} > 1$ and $\liminf_t q_t^{\beta} > 1$. Then, there exist $\eta, \mu > 0$ such that $q_s^{\beta} \geq 1 + \eta$ and $q_t^{\beta} \geq 1 + \mu$ for all s, t, which implies that

$$\frac{h_{st}}{\ell_{st}} \geq \frac{\eta\mu}{(1+\eta)(1+\mu)} \Rightarrow \frac{h_{st}^{\beta}}{\ell_{st}^{\beta}} \geq \frac{\eta^{\beta}\mu^{\beta}}{(1+\eta)^{\beta}(1+\mu)^{\beta}}.$$

For every $\xi > 0$ and each $y \in Y$, we have

$$\begin{aligned}
\frac{1}{\ell_{st}^{\beta}} & \left|\left\{(m,n): m \leq j_s, n \leq k_t, \left|\rho_y\left(\frac{C_{\sigma^m(u)\sigma^n(v)}}{D_{\sigma^m(u)\sigma^n(v)}}\right) - \lambda\right| \geq \xi\right\}\right| \\
& \geq \frac{1}{\ell_{st}^{\beta}} \left|\left\{(m,n) \in I_{st}: \left|\rho_y\left(\frac{C_{\sigma^m(u)\sigma^n(v)}}{D_{\sigma^m(u)\sigma^n(v)}}\right) - \lambda\right| \geq \xi\right\}\right| \\
& = \frac{h_{st}^{\beta}}{\ell_{st}^{\beta}} \frac{1}{h_{st}^{\beta}} \left|\left\{(m,n) \in I_{st}: \left|\rho_y\left(\frac{C_{\sigma^m(u)\sigma^n(v)}}{D_{\sigma^m(u)\sigma^n(v)}}\right) - \lambda\right| \geq \xi\right\}\right| \\
& \geq \frac{\eta^{\beta}\mu^{\beta}}{(1+\eta)^{\beta}(1+\mu)^{\beta}} \frac{1}{h_{st}^{\beta}} \left|\left\{(m,n) \in I_{st}: \left|\rho_y\left(\frac{C_{\sigma^m(u)\sigma^n(v)}}{D_{\sigma^m(u)\sigma^n(v)}}\right) - \lambda\right| \geq \xi\right\}\right|,
\end{aligned}$$

for all u, v. If $C_{mn} \overset{W_2^{\lambda}(S_{\sigma}^{\beta})}{\sim} D_{mn}$, then for each $y \in Y$ the term on the left side of the above inequality convergent to 0 and this implies that

$$\frac{1}{h_{st}^{\beta}} \left|\left\{(m,n) \in I_{st}: \left|\rho_y(\frac{C_{\sigma^m(u)\sigma^n(v)}}{D_{\sigma^m(u)\sigma^n(v)}}) - \lambda\right| \geq \xi\right\}\right| \to 0$$

uniformly in u, v. Thus, we get $C_{mn} \overset{W_2^{\lambda}(S_{\sigma\theta}^{\eta})}{\sim} D_{mn}$.

Theorem 2. If $\limsup_s q_s < \infty$ and $\limsup_t q_t < \infty$, then

$$C_{mn} \overset{W_2^{\lambda}(S_{\sigma\theta}^{\beta})}{\sim} D_{mn} \Rightarrow C_{mn} \overset{W_2^{\lambda}(S_{\sigma}^{\beta})}{\sim} D_{mn},$$

where $0 < \beta \leq 1$.

Proof. Let $\limsup_s q_s < \infty$ and $\limsup_t q_t < \infty$. Then, there exist $M, N > 0$ such that $q_s < M$ and $q_t < N$ for all s, t. Also, we suppose that $C_{mn} \overset{W_2^{\lambda}(S_{\sigma\theta}^{\beta})}{\sim} D_{mn}$ (where $0 < \beta \leq 1$) and $\xi > 0$, and let

$$\kappa_{st} := \left|\left\{(m,n) \in I_{st}: \left|\rho_y\left(\frac{C_{\sigma^m(u)\sigma^n(v)}}{D_{\sigma^m(u)\sigma^n(v)}}\right) - \lambda\right| \geq \xi\right\}\right|.$$

Then, there exist $s_0, t_0 \in \mathbb{N}$ such that for every $\xi > 0$, each $y \in Y$ and all $s \geq s_0, t \geq t_0$

$$\frac{\kappa_{st}}{h_{st}^{\beta}} < \xi,$$

for all u, v. Now, let

$\gamma := \max\{\kappa_{st} : 1 \leq s \leq s_0, 1 \leq t \leq t_0\}$,

and let p and q be any integers satisfying $j_{s-1} < p \leq j_s$ and $k_{t-1} < q \leq k_t$. Then, for each $y \in Y$ we have

$$\frac{1}{(pq)^\beta}\left|\left\{(m,n): m \leq p, n \leq q, \left|\rho_y\left(\frac{C_{\sigma^m(u)\sigma^n(v)}}{D_{\sigma^m(u)\sigma^n(v)}}\right) - \lambda\right| \geq \xi\right\}\right|$$

$$\leq \frac{1}{\ell^\beta_{(s-1)(t-1)}}\left|\left\{(m,n): m \leq j_s, n \leq k_t, \left|\rho_y\left(\frac{C_{\sigma^m(u)\sigma^n(v)}}{D_{\sigma^m(u)\sigma^n(v)}}\right) - \lambda\right| \geq \xi\right\}\right|$$

$$= \frac{1}{\ell^\beta_{(s-1)(t-1)}}\{\kappa_{11} + \kappa_{12} + \kappa_{21} + \kappa_{22} + \cdots + \kappa_{s_0t_0} + \cdots + \kappa_{st}\}$$

$$\leq \frac{s_0t_0}{\ell^\beta_{(s-1)(t-1)}}\left(\max_{\substack{1\leq m\leq s_0\\1\leq n\leq t_0}}\{\kappa_{mn}\}\right) + \frac{1}{\ell^\beta_{(s-1)(t-1)}}\left\{h^\beta_{s_0(t_0+1)}\frac{\kappa_{s_0(t_0+1)}}{h^\beta_{s_0(t_0+1)}}\right.$$

$$\left. + h^\beta_{(s_0+1)t_0}\frac{\kappa_{(s_0+1)t_0}}{h^\beta_{(s_0+1)t_0}} + h^\beta_{(s_0+1)(t_0+1)}\frac{\kappa_{(s_0+1)(t_0+1)}}{h^\beta_{(s_0+1)(t_0+1)}} + \cdots + h^\beta_{st}\frac{\kappa_{st}}{h^\beta_{st}}\right\}$$

$$\leq \frac{s_0t_0\gamma}{\ell^\beta_{(s-1)(t-1)}} + \frac{1}{\ell^\beta_{(s-1)(t-1)}}\left(\sup_{\substack{s>s_0\\t>t_0}}\frac{\kappa_{st}}{h^\beta_{st}}\right)\left(\sum_{m,n\geq s_0,t_0}^{s,t} h^\beta_{mn}\right)$$

$$\leq \frac{s_0t_0\gamma}{\ell^\beta_{(s-1)(t-1)}} + \frac{1}{\ell_{(s-1)(t-1)}}\left(\sup_{\substack{s>s_0\\t>t_0}}\frac{\kappa_{st}}{h^\beta_{st}}\right)\left(\sum_{m,n\geq s_0,t_0}^{s,t} h_{mn}\right)$$

$$\leq \frac{s_0t_0\gamma}{\ell^\beta_{(s-1)(t-1)}} + \xi\frac{(j_s - j_{s_0})(k_t - k_{t_0})}{\ell_{(s-1)(t-1)}}$$

$$\leq \frac{s_0t_0\gamma}{\ell^\beta_{(s-1)(t-1)}} + \xi q_s q_t$$

$$\leq \frac{s_0t_0\gamma}{\ell^\beta_{(s-1)(t-1)}} + \xi MN,$$

for all u, v. Since $j_{s-1}, k_{t-1} \to \infty$ as $p, q \to \infty$, it follows that for each $y \in Y$

$$\frac{1}{(pq)^\beta}\left|\left\{(m,n): m \leq p, n \leq q, \left|\rho_y\left(\frac{C_{\sigma^m(u)\sigma^n(v)}}{D_{\sigma^m(u)\sigma^n(v)}}\right) - \lambda\right| \geq \xi\right\}\right| \to 0$$

uniformly in u, v. Thus, we get $C_{mn} \overset{W_2^\lambda(S_\sigma^\beta)}{\sim} D_{mn}$.

Theorem 3. If

$1 < \liminf_s q_s^\beta \leq \limsup_s q_s < \infty$ and $1 < \liminf_t q_t^\beta \leq \limsup_t q_t < \infty$, where $0 < \beta \leq 1$, then

$$C_{mn} \overset{W_2^\lambda(S_{\sigma\theta}^\beta)}{\sim} D_{mn} \Leftrightarrow C_{mn} \overset{W_2^\lambda(S_\sigma^\beta)}{\sim} D_{mn}.$$

Proof. This can be obtained from Theorem 1 and Theorem 2, immediately.

Theorem 4. If $\liminf_{s,t\to\infty} \frac{h_{st}^\beta}{\ell_{st}} > 0$ where $0 < \beta \leq 1$, then $\{W_2^\lambda(S_\sigma)\} \subseteq \{W_2^\lambda(S_{\sigma\theta}^\beta)\}$.

Proof. For every $\xi > 0$ and each $y \in Y$, it is obvious that

$$\left\{(m,n): m \leq j_s, n \leq k_t, \left|\rho_y\left(\frac{C_{\sigma^m(u)\sigma^n(v)}}{D_{\sigma^m(u)\sigma^n(v)}}\right) - \lambda\right| \geq \xi\right\}$$
$$\supset \left\{(m,n) \in I_{st}: \left|\rho_y\left(\frac{C_{\sigma^m(u)\sigma^n(v)}}{D_{\sigma^m(u)\sigma^n(v)}}\right) - \lambda\right| \geq \xi\right\}.$$

Thus, we have

$$\frac{1}{\ell_{st}}\left|\left\{(m,n): m \leq j_s, n \leq k_t, \left|\rho_y\left(\frac{C_{\sigma^m(u)\sigma^n(v)}}{D_{\sigma^m(u)\sigma^n(v)}}\right) - \lambda\right| \geq \xi\right\}\right|$$
$$\geq \frac{1}{\ell_{st}}\left|\left\{(m,n) \in I_{st}: \left|\rho_y\left(\frac{C_{\sigma^m(u)\sigma^n(v)}}{D_{\sigma^m(u)\sigma^n(v)}}\right) - \lambda\right| \geq \xi\right\}\right|$$
$$= \frac{h_{st}^\beta}{\ell_{st}} \frac{1}{h_{st}^\beta}\left|\left\{(m,n) \in I_{st}: \left|\rho_y\left(\frac{C_{\sigma^m(u)\sigma^n(v)}}{D_{\sigma^m(u)\sigma^n(v)}}\right) - \lambda\right| \geq \xi\right\}\right|,$$

for all u, v. If $C_{mn} \overset{W_2^\lambda(S_\sigma)}{\sim} C$, then for each $y \in Y$ the term on the left side of the above inequality convergent to 0 and this implies that

$$\frac{1}{h_{st}^\beta}\left|\left\{(m,n) \in I_{st}: \left|\rho_y\left(\frac{C_{\sigma^m(u)\sigma^n(v)}}{D_{\sigma^m(u)\sigma^n(v)}}\right) - \lambda\right| \geq \xi\right\}\right| \to 0$$

uniformly in u, v. Thus, we get $C_{mn} \overset{W_2^\lambda(S_{\sigma\theta}^\eta)}{\sim} C$. Consequently, $\{W_2^\lambda(S_\sigma)\} \subseteq \{W_2^\lambda(S_{\sigma\theta}^\beta)\}$.

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