







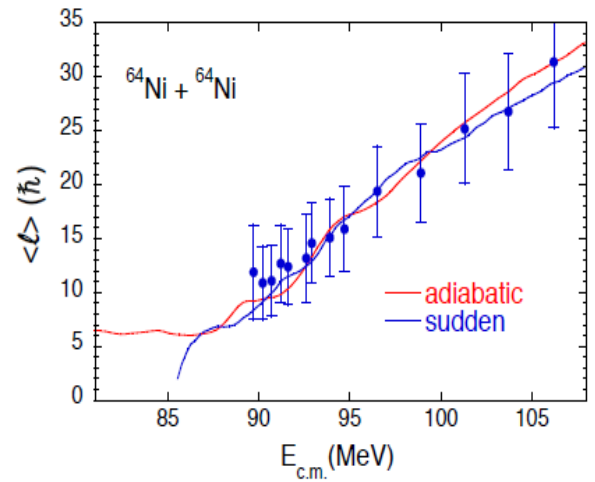


Apart from this, the Pauli principle can also be interpreted in terms of Pauli attraction rather than Pauli repulsion, although that it is generally believed that it results in a repulsive core, as was the case in the sudden model [5,6]. Notice that Pauli principle aims to quench the radial wave function at short distances and that the Pauli repulsion and the Pauli attraction both accomplish this similarly. Ohkubo [14] has recently put forward this point for a nucleon-nucleon interaction, supporting the results in [11–13]. The corresponding Pauli attraction naturally implies that the potential must be deep enough to hold the prohibited states, as in the case of supersymmetric partner potentials, as the physically relevant wave function must be orthogonalized to Pauli forbidden states. This point makes a link between the potentials used by sudden and adiabatic potentials.

This short discussion reveals a possibility regarding a hidden supersymmetric relation between the sudden and adiabatic model analysis of fusion observables for heavy ion interactions. Considering the work in [13], which investigates wave function-sensitive properties of the supersymmetric potentials by a halo transfer reaction, it is helpful to remind that a deep potential and its phase equivalent shallow partner, that are used for building the related entrance and exit channel wave functions for the given fusion process, are constructed with identical phase shifts so that any differences in physical quantities, such as relevant cross-section analyses in reactions, can only be explained by differences in the corresponding wave functions of the partner potentials. Along this line, the researchers in [13] observed that reconstructed PEP have led to relative motion wave functions very similar to those generated by the deep potentials outside the core region, but with no radial node at small distance due to the singularity of the shallow partner potential. As a result, it is anticipated that both types of potentials will exhibit somehow different off-shell behaviours and produce results that are qualitatively different. However, interestingly, no major difference was found between the *rms* radii calculated from these quite different two-body interactions. Considering transfer reaction  $^{11}\text{Be}(p,d)^{10}\text{Be}$ , a follow-up study in Ref. [13] on the effects of utilizing such phase equivalent two-body potentials to characterize weakly bound  $^{11}\text{Be}$  and deuteron nuclei in three-body model calculations, as entrance and exit channel wavefunction components, yielded curiously similar results. Hence, the authors of [13] concluded that the short-range behaviour of the corresponding wave functions for the deep and phase equivalent shallow potentials, which coincide at large distances but differ at small distances by the additional node appearing inside the core, is not significant for the analysis of such transfer reaction observables.

Given the above-discussed research findings, and with the expertise gained from the work in [13], we thus propose that, despite the structural differences between the potential functions used in these analytical treatment techniques, there may be a similar relationship between the potentials used by sudden and adiabatic models. This could be the reason for the remarkable similarities between the analyses of fusion observables for heavy-ion systems seen in Fig. 5.

To justify our entire debate here, one may also consider the mean angular momentum calculations of the compound nucleus, illustrated in Fig.7, performed by sudden and adiabatic models.



**Fig.7** Average angular momentum of compound nucleus vs. incident energy for  $^{64}\text{Ni} + ^{64}\text{Ni}$ . Taken from [15]. The results of the sudden model were performed using the M3Y+ repulsive potential.

This physical quantity is evaluated in a way [16] that

$$\langle \ell \rangle (E) = \frac{\frac{\pi}{k^2} \sum_{\ell} \ell (2\ell + 1) T_{\ell}}{\frac{\pi}{k^2} \sum_{\ell} (2\ell + 1) T_{\ell}} \quad (1)$$

in which  $\sigma_{\ell}(E) = (\pi/k^2)(2\ell+1)T_{\ell}(E)$  is the partial wave cross section leading to the total excitation function,  $\sigma_{fus}(E) = \sum_{\ell} \sigma_{\ell}(E)$ , and  $T_{\ell}$  is the quantum-mechanical transmission probability through the potential barrier. The radial wave function  $u_{\ell}(r)$  in the exit channel for the partial wave  $\ell$  is in relation with  $T_{\ell}$  as given by [3,4,16]

$$u_{\ell}(r) = \sqrt{\frac{k}{k_{\ell}(r)}} T_{\ell} \exp\left(-i \int_r^{r_{abs}} k_{\ell}(r') dr'\right) \quad (r \leq r_{abs}) \quad (2)$$

where the integral region defines inside of the Coulomb barrier and  $k_\ell(r)$  is the local wave number

$$k_\ell(r) = \sqrt{\frac{2\mu}{\hbar^2}(E - U(r))} ,$$

$$U(r) = V_N(r) + V_C(r) + \frac{\ell(\ell + 1)\hbar^2}{2\mu r^2} \quad (3)$$

which takes into account the real part of the full internuclear potential,  $U(r)$ , and  $r_{abs}$  is the absorption radius, together with  $k$  being the incident wave number ( $= 2\mu E/\hbar^2$ ). Obviously,  $\mu$  and  $E$  are the reduced mass and the incident energy in the center of mass frame for the reaction of interest. With this consideration, it is clearly seen that the effects of the related terms  $T_\ell$ ,  $U(r)$  and  $u(r)$  in Eqs. (2) and (3) are not remarkably large in the calculated physical observables, as shown in Figs. 5 and 7, justifying that the difference in potential and wavefunction behavior that appeared in these treatments does not cause a significant distinction in the theoretical reproduction of physical quantities at low energies similar to the work in [13]. This point once more reveals a prospective interconnection between the sudden and adiabatic models in terms of their potential descriptions used for analysing fusion reactions at low energies.

For a closing remark, we draw the attention of the reader to the most recently published [17] an impressived work in which the curvature of the potential barrier,  $\hbar w$ , has been modified as  $\hbar w \rightarrow (\hbar w) \exp\left[\lambda \frac{E-V}{V}\right]$  and shown that the modified Wong formula with this novel term reproduces fusion cross sections quite well for various systems, involving heavy ion interactions, across the whole energy range including fusion hindrance phenomenon, unlike the original expression. Considering the analysis in [17] one can clearly see that the deficiency, being the insensitivity of barrier properties such as radius ( $R$ ), height ( $V$ ) and curvature ( $\hbar w$ ) to the angular momentum, in the original Wong formula has been removed through the modification of only the curvature term. The work in [17] thus demonstrates that  $\ell$  –dependence of  $R$ ,  $V$  and  $\hbar w$  may be correlated and can be simulated with a single  $\ell$  –dependence of the  $\hbar w$  term. At this stage, remembering also the  $\ell$  –dependency of a shallow phase equivalent partner potential due to elimination of unphysical Pauli forbidden state(s), it is not hard to establish a connection to the shallow nature of the potential behaviour in the sudden model because of the same reason: Pauli repulsion between the fermions of reacting nuclei inside the barrier. This plausible connection seems another evidence for justifying the whole discussion in this section.

#### 4. Concluding Remarks

Although the physical recipe behind the hindrance is given differently through the sudden and the adiabatic treatment technique, both emphasize the importance of dynamical effects within overlapping regions. Overall, it is not explicitly accepted that the hindrance phenomenon is better described with a two-body potential producing a shallow potential pocket, or if an adiabatic approach is more appropriate due to its different potential structure consideration. Discriminating between these two models will require challenging measurements. We believe that this simple and intuitive discussion through this paper would shed light on the related area.

#### Conflict of Interest

The authors have no conflict of interest.

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