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New Analytical Solutions of Fractional Complex Ginzburg-Landau Equation

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Abstract

In recent years, nonlinear concepts have attracted a lot of attention due to the deep mathematics and physics they contain. In explaining these concepts, nonlinear differential equations appear as an inevitable tool. In the past century, considerable efforts have been made and will continue to be made to solve many nonlinear differential equations. This study is also a step towards analytical solution of the complex Ginzburg-Landau equation (CGLE) used to describe many phenomena on a wide scale. In this study, the CGLE was solved analytically by $(1/G')$-expansion method.

1. Introduction

In recent years mathematical and physical aspects of nonlinear phenomena draw much attention [1–4]. Since true laws of nature are drawn by nonlinear interactions. And, one of the inevitable tools for translating these laws into a mathematical language are nonlinear differential equations. It would not be an exaggeration to say that the past century was a century of nonlinear equations. Many different nonlinear differential equations have been the subject of studies to explain various nonlinear phenomena. Some of the most famous of these equations are Korteweg - de Vries (KdV) [5], Boussinesq [6], Cahn-Hilliard [7], nonlinear Schrödinger [8] and Ginzburg- Landau [9], etc. Especially complex form of Ginzburg-Landau equation (CGLE) is very interesting due to its capability of explaining very complex events in physics such as superconductivity, superfluidity [10], strings in field theory [11], Bose-Einstein condensation [12], etc. Due to its flexibility CGLE has been studied extensively by physicist and mathematicians.

In recent years, analytical and numerical solutions of fractional differential partial differential equations have been obtained by different methods [7, 13–16]. $(1/G')$-expansion method has been widely used to obtain analytical solutions of partial differential equations [17–19]. This method stands out for its flexibility, reliability and convenience. In this study, new wave solutions of conformable time fractional CGLE were obtained by $(1/G')$-expansion method.

2. Governing Equation

In this study, conformable time fractional CGLE is taken account as the governing equation which is in the form of;

$$iD_t^{\eta}+aq_{xx}+bG(|q|^2)q=\frac{1}{|q|^2q^*}\left[\alpha|q|^2(|q|^2)_{xx}-\beta\{(|q|^2)_x\}^2\right]+\gamma q \tag{2.1}$$

where $\eta\in(0,1)$, x represents the distance along the fibers, while t represents the time; a,b,α,β and γ are constants. The coefficients a and b arrise from the group velocity dispersion (GVD) and nonlinearity. The terms α,β and γ arrise from the perturbation effects in particular, γ occurs from the debasement effect. In Eq. (2.1), function G must possess the uniformity of the complex function $G(|q|^2)q$ which is k times continuously differentiable, consequently

$$G(|q|^2)q\in\bigcup_{m,n=l}^{\infty}C^k((-n,n)\times(-m,m);R^2). \tag{2.2}$$

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To obtain the solution of Eq. (2.1), the usual decomposition into phase-amplitude components produces:

$$q(x,t) = H(\xi)e^{(-\kappa x+\omega \frac{t^\eta}{\eta}+\theta)} \tag{2.3}$$

where the ξ is defined as

$$\xi = k(x - v\frac{t^\eta}{\eta}). \tag{2.4}$$

The function H denotes the pulse shape, v implies the speed of the soliton, κ denotes the soliton frequency; ω represents the soliton wave number, θ represents the phase constant. When the amplitude-phase decomposition subrogated into Eq. (2.1) and separating into real and imaginary parts, the following equations yields:

$$-\omega H + a(k^2 H_{\xi\xi} - \kappa^2 H) + bG(H^2)H = 2k^2(\alpha - 2\beta)\frac{(H_\xi)^2}{H} + 2k^2\alpha H_{\xi\xi} + \gamma H \tag{2.5}$$

and

$$v = -2a\kappa.$$

By decides on

$$\alpha = 2\beta$$

the first term on the right-hand side of Eq. (2.5) set to zero. Thus Eq. (2.1) becomes

$$iD_t^\eta + aq_{xx} + bG(|q|^2)q = \frac{\beta}{|q|^2 q^*}\left[2|q|^2(|q|^2)_{xx} - \{(|q|^2)_x\}^2\right] + \gamma q \tag{2.6}$$

and Eq. (2.5) becomes

$$k^2(a - 4\beta)H_{\xi\xi} - (\omega + a\kappa^2 + \gamma)H + bG(H^2)H = 0. \tag{2.7}$$

3. $(1/G')$-Expansion Method

The $(1/G')$-expansion method is implemented to various partial differential equations (PDEs) [17–19]. This method is a powerful analtical method for the computation of analytical solutions of PDEs. Now, lets deal with the nonlinear conformable time fractional partial differential equation for $\varphi(x,t)$ in the form

$$H\left(\varphi, \frac{\partial^\eta \varphi}{\partial t^\eta}, \frac{\partial \varphi}{\partial x}, \frac{\partial^2 \varphi}{\partial t^2}, \frac{\partial^2 \varphi}{\partial x^2}, \dots\right) = 0 \tag{3.1}$$

where $\varphi(x,t)$ is the unknown function and H is the polynomial of $\varphi(x,t)$ and its partial derivatives.
Presentation the wave variable as

$$\varphi(x,t) = \varphi(\xi), \xi = k(x - v\frac{t^\eta}{\eta}). \tag{3.2}$$

where k and c are parameters. Using Eq. (3.2), we get Eq. (3.1) becomes an ordinary differential equation for $\varphi = \varphi(\xi)$

$$F\left(\varphi, \varphi', \varphi'', \varphi''', \dots\right) = 0. \tag{3.3}$$

where prime implies derivative respect to ξ. According to $(1/G')$-expansion method, it is supposed that the analytical solutions of Eq. (3.3) can be expressed as a polynomial of $(1/G')$ as

$$\varphi(\xi) = \sum_{i=0}^{n} a_i \left(\frac{1}{G'}\right)^i, \quad a_n \neq 0 \tag{3.4}$$

where $G = G(\xi)$ satisfies the second order ordinary differential equation

$$G'' + \lambda G' + \mu = 0 \tag{3.5}$$

and $a_i (i = 0, \dots, n), \lambda, \mu$ are constants to be determined later. To obtain the solution of Eq. (3.5) with $G = G(\xi)$, the Eq. (3.4) will contain the following equation

$$\frac{1}{G'(\xi)} = \frac{1}{-\frac{\mu}{\lambda} + A\tanh(\lambda\xi) - A\sinh(\lambda\xi)} \tag{3.6}$$

where A is integral constant.
Step1.
The positive integer n in Eq. (3.4) can be stated by figuring out the homogeneous balance between the highest order derivatives and the highest nonlinear terms of $\varphi(\xi)$ in Eq. (3.3).
Step2.
Replacing (3.4) with Eq. (3.5) into Eq. (3.3) and simplifying by collecting together all the same powered terms of $(1/G')$, the left hand side of Eq. (3.3) is turns into a polynomial. After equalizing each coefficient of this polynomial to zero, we get a set of algebraic equations in terms of $a_i (i = 0, \dots, n), \lambda, \mu, c, k$.
Step3.
By solving the system by symbolic computer software, then replacing the results with the solutions of Eq. (3.5) into Eq. (3.4) leads to analytical solutions of Eq. (3.3).

4. Analytical Solutions of Complex Ginzburg-Landau Equation

As it can be seen Kerr law nonlinearity can be applied to Eq. (4.2). Since, non-harmonic motion of electrons with an external electric field cause to nonlinear responses in the optical fiber. Due to the Kerr law nonlinearity, $G(u)$ is can be taken as u, hence Eq. (2.6) becomes

$$iD_t^{\eta}+aq_{xx}+b|q|^2q=\frac{\beta}{|q|^2q^*}\left[2|q|^2(|q|^2)_{xx}-\{(|q|^2)_x\}^2\right]+\gamma q \tag{4.1}$$

and Eq. (4.2) turns into

$$k^2(a-4\beta)H_{\xi\xi}-(\omega+a\kappa^2+\gamma)H+bH^3=0. \tag{4.2}$$

According to the balance principle, we obtain $n=1$. Consequently, the analytical solution of Eq.(4.2) can be obtanied as

$$H(\xi)=a_0+a_1\left(\frac{1}{G'}\right). \tag{4.3}$$

and thus

$$\varphi''(\xi)=2a_1\mu^2\left(\frac{1}{G'(\xi)}\right)^3+3a_1\lambda\mu\lambda\left(\frac{1}{G'(\xi)}\right)^2+a_1\lambda^2\left(\frac{1}{G'(\xi)}\right). \tag{4.4}$$

By replacing Eqs. (4.3)-(4.4) by Eq. (4.2) and gathering together all the same powered terms of $(1/G')$, the left hand side of Eq.(4.2) is turned into another polynomial in $(1/G')$. Equalizing each coefficient of this polynomial to zero, gets a set of algebraic equations as follows:

$$\begin{aligned}
&\left(\frac{1}{G'(\xi)}\right)^0: -a\kappa^2a_0-\omega a_0+ba_0^3-\gamma a_0=0,\\
&\left(\frac{1}{G'(\xi)}\right)^1: -4k^2\beta a_1\lambda^2-\omega a_1-a\kappa^2a_1-\gamma a_1+3ba_0^2a_1+k^2aa_1\lambda^2=0,\\
&\left(\frac{1}{G'(\xi)}\right)^2: 3k^2aa_1\lambda\mu-12k^2\beta a_1\lambda\mu+3ba_0a_1^2=0,\\
&\left(\frac{1}{G'(\xi)}\right)^3: -8k^2\beta a_1\mu^2+2k^2aa_1\mu^2+ba_1^3=0.
\end{aligned} \tag{4.5}$$

Solving the system above, gets

$$k=\pm\frac{\sqrt{2a\kappa^2+2\gamma+2\omega}}{\lambda\sqrt{4\beta-a}},\quad a_0=\pm\frac{\sqrt{a\kappa^2+\gamma+\omega}}{\sqrt{b}},\quad a_1=\pm\frac{2\mu\sqrt{a\kappa^2+\gamma+\omega}}{\sqrt{b}\lambda}. \tag{4.6}$$

By the help of the statements (4.6), (4.3) and (3.6), we obtain analytical solutions of Eq. (4.1) as follows:

$$q_{1,2}(x,t)=\pm\frac{\sqrt{a\kappa^2+\gamma+\omega}}{\sqrt{b}}\left(\frac{2\mu}{\lambda\left(\mp A\sinh(\delta\xi)+A\cosh(\delta\xi)-\frac{\mu}{\lambda}\right)}+1\right)e^{i\left(\theta+\frac{\omega t^{\eta}}{\eta}-\kappa x\right)}$$

where

$$\xi=\frac{2a\kappa t^{\eta}}{\eta}+x$$

and

$$\delta=\frac{\sqrt{2a\kappa^2+2\gamma+2\omega}}{\sqrt{4\beta-a}}.$$

5. Conclusions

In this study, the complex Ginzburg-Landau equation (CGLE) used in the evaluation of many physical phenomena such as Bose-Einstein condensation, superconductivity, super-fluidity, semiconductor laser excitations was solved analytically. In order to solve the CGLE, the $(1/G')$-expansion method, which mathematicians have being used in analytical solution of nonlinear partial differential equations in recent years, has been used. In this study, it has been shown that the $(1/G')$-expansion method can be successfully applied in the analytical solutions of the CGLE. In addition, the flexibility, reliability and convenience of the method have been demonstrated with a new study.

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